Oskar Klein, the sixth dimension and the strength of a magnetic pole

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This work extends to six dimensions the idea first proposed by Klein regarding a closed space in the context of a fifth dimension and its link to quantum theory. The main result is a formula that expresses the value of the characteristic length of the sixth dimension in terms of the strength of a magnetic monopole g. It is shown that in the case of Dirac's monopole, the ratio of the characteristic lengths of the fifth and sixth dimension corresponds to twice the fine structure constant α. Possible consequences of the idea are discussed.

I. INTRODUCTION

It is well known that the world lines of charged particles in the presence of gravitational and electromagnetic fields can be viewed as geodesics in the 5D space-time first proposed by T. Kaluza [1]. Kaluza's treatment was later improved by Klein in 1926 [2], in a quantum theoretical approach, giving rise to the so-called Kaluza-Klein theory [3]. This formalism also makes possible to propose, based on plausible grounds, a simple formula that gives the value of the characteristic length of the fifth dimension in terms of fundamental constants of nature. The original Kaluza-Klein theories have been, since the 1920's, the subject of further analysis that include generalizations to a larger number of dimensions [4], cosmological implications [5] and magnetic monopoles [6], most of them treated *a-la* Dirac [7].

This last subject deserves more attention. Recently [8], the authors have observed that a simple 6D generalization of Kaluza's 5D metric leads naturally to the geodesics of particles possessing (hypothetical) fundamental magnetic charges. Further, this 6D space-time reproduces Maxwell's equations in the presence of magnetic monopoles and allows the establishment of a wave equation for the vector potential [9]. In view of these facts, we have reanalyzed Klein's formalism in order to establish a link between the eigenvalue of the sixth dimensional momentum operator and the discrete nature of the hypothetical point charges. We also examine the characteristic length of the sixth dimension by setting the magnetic charge equal to the value proposed by Dirac.

II. KALUZA'S CLASSIC THEORY IN 6 DIMENSIONS

We start this section observing that the geodesic equation

$$\frac{d^2x^\alpha}{dt^2} + \Gamma^\alpha_{\beta\lambda}\frac{dx^\beta}{dt}\frac{dx^\lambda}{dt} = 0 \tag{1}$$

contains the equation of motion of a magnetic charged particle in the presence of a *static* electromagnetic field given that:

$$x^\alpha = \begin{bmatrix} x^1 \\ x^2 \\ x^3 \\ ct \\ 0 \\ \frac{1}{c\xi}\frac{g}{m}t \end{bmatrix} \tag{2}$$

$$\dot{x}^\alpha = \frac{dx^\alpha}{dt} \tag{3}$$

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -\xi\varphi & -\frac{\xi}{c}\eta \\ 0 & 0 & 0 & -\xi\varphi & 1 & 0 \\ 0 & 0 & 0 & -\frac{\xi}{c}\eta & 0 & 1 \end{bmatrix} \quad (4)$$

In Eq. (1), the indices α, β and γ run from 1 to 6 and

$$\xi = \sqrt{\frac{16\pi G\epsilon_0}{c^2}},$$

is introduced in order to obtain the correct gravitational and electromagnetic static limits (Poisson equations) as shown in the appendix of Ref. [10]. Also, in Eqs. (2-4), $\varepsilon_o = 8.85 \times 10^{-12}\frac{c^2}{N\,m^2}$, $G = 6.67 \times 10^{-11}\frac{N\,m^2}{kg^2}$ and $c = 3 \times 10^8\frac{m}{s}$. Dimensional consistency requires a magnetic charge g measured in $\frac{m}{s}c$. The metric (4) involves the static potentials φ and η, and can easily be extended to include non-static fields [1]-[8]. Nevertheless, for the ideas here presented, the static fields metric will be enough.

III. KLEIN'S HYPOTHESIS APPLIED TO x^6

Following Klein [3], we write for the six dimensional momentum the equation:

$$p^6 = m_g\dot{x}^6 = \frac{g}{c\xi} \quad (5)$$

If we now apply the standard quantization rules to the quantity defined in Eq. (5), then:

$$\frac{g}{c\xi} = \frac{N_g h}{\ell_6} \quad (6)$$

where N_g is an integer number, as required by the quantization of charge, and ℓ_6 represents the characteristic length of the (compact) periodic six dimension. Use has been made of Eqs. (2-3). Equation (6) is a natural extension of Klein's arguments regarding the sixth dimension and expresses the idea that its characteristic length is inversely proportional to the value of the magnetic monopole strength g.

IV. CHARACTERIZATION OF THE SIXTH DIMENSION

There is no experimental evidence to fix the two parameters in Eq. (6): the characteristic length ℓ_6 and the strength of the magnetic charge g. In this sense we propose two alternatives. Firstly, an assumption can be made about the size of the sixth dimension compared to ℓ_5 which is given by

$$\ell_5 = \frac{h}{e}\sqrt{\frac{16\pi G\varepsilon_0}{c^2}} \quad (7)$$

If both compact dimensions posses the same characteristic length, then g is simply given by $g = qc$. If otherwise, we assume a vanishing value for g, then ℓ_6 would become infinite. This idea seems consistent with the existence of non-compact extra dimensions. If compactness is a property sought for dimensional spaces of $d > 4$, the existence of the still undetected magnetic monopole would favour the idea.

The second alternative is to give a value for g which fixes the length ℓ_6. Most work regarding Kaluza-Klein theory in the context of magnetic monopoles is based on Dirac's pioneering work [7]. In that context, the magnetic monopole possesses a charge $g = \frac{\varepsilon_o h c^2}{e} = 3.3 \times 10^{-9} C\frac{m}{s}$ and thus,

$$\ell_6 = \frac{e}{\varepsilon_o c^2}\sqrt{16\pi G\varepsilon_o} \quad (8)$$

In this case, by fixing the magnitude of the magnetic charge, one can determine the size of the sixth dimension which is smaller than the fifth dimension as determined by Klein. Moreover, the ratio between the two extra dimensions is found to be:

$$\frac{\ell_6}{\ell_5} = \frac{e^2}{h\varepsilon_o c} = 2\alpha \tag{9}$$

where α is the fine structure constant, which has been found to be related with the length of the extra dimensions in the Kaluza-Klein context by other authors [10].

V. SUMMARY AND DISCUSSION

In previous work, we've shown that Kaluza's theory can be extended to six dimensions in order to include an elementary magnetic charge in electromagnetic theory. By doing so, Maxwell's equations become symmetric and can be derived directly from Einstein's field equations. Here, the theory formulated in Ref. [8] has been complemented by applying the quantization hypothesis proposed by Klein for the fifth dimension to the new sixth dimension. As a result, an expression for the characteristic length of the sixth dimension is expressed in terms of the strength of the magnetic charge [see Eq. (8)]. In order to estimate either of these quantities we propose two alternatives. If ℓ_6 is compared directly with ℓ_5, a value for the magnetic charge is obtained. On the other hand, if g is assumed to correspond to the value of Dirac's monopole, the size of the sixth dimension that can be obtained is related with ℓ_5 through the fine structure constant.

Most work extending Kaluza's ideas is based on Gauge theories and topological arguments. The results here presented arise from a simpler methodology in which the dimensionality of space-time is increased in the same way Kaluza did in his classic paper. Hence, Klein's hypothesis can be applied directly to the extra dimension and thus, the analysis can be done in a straightforward and simple manner.

Kaluza's idea has been abandoned for years for various reasons, one of them being the mass spectra of the particles obtained in five dimensions [12]. In this sense we wish to point out that the topology in 6 dimensions may differ with the one in which that calculation is based. Calculating the mass spectra in a new topology is a task worth pursuing since, as some drawbacks of Kaluza's classical theory were surmounted by extending the metric in Ref. [8], new elements may enter the calculation for the mass spectra and modify the results whereas more symmetry is achieved.

It is the opinion of the authors that physical ideas leading to simple measurable properties of spaces of more than four dimensions must be pursued. Part of this work has already been introduced in the form of a Kaluza-Klein magnetohydrodynamical framework [10, 13]. We believe that the present work is another step in this direction.

This work has been supported by CONACyT (Mexico), project 41081-F and FICSAC (Mexico), PFSA.

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